Stabilization and Robustness Analysis for a Chain of Saturating Integrators with Imprecise Measurements

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***Abstract*— We solve a challenging input-to-state stabilization problem for a chain of saturated integrators when the variables are not accurately measured. We use a recent backstepping approach with pointwise delays and no distributed terms.**

***Index Terms*— Stability, backstepping, saturations.**

I. Introduction

The difficult problem of stabilizing systems with bounded controls has a long history. To solve it, semi-global state and output feedback stabilization results [22] have been obtained via linear control laws inside saturations. Also, crucial regional [3] stability results for some linear and nonlinear systems (based on LMI techniques [2], [21]) were shown in [20]. Forwarding and bounded backstepping provide globally asymptotically stabilizing control laws for some nonlinear systems. Bounded backstepping was developed in [12] and forwarding has been studied, e.g., in [16] and [19].

A significantly different backstepping design was proposed in [13] and [14]. It used artificial pointwise delays in the control, which circumvent the problem of determining Lie derivatives of the fictitious controls. This relaxes the smoothness requirement that was imposed on the fictitious control in all previous contributions on backstepping. Moreover, for many systems of feedback or feedforward form, it can be used to determine globally asymptotically stabilizing bounded controls that are simpler than those of [12].

The advantages of [13] motivate the present work, which adapts the approach from [13] to a control problem for a chain of saturating integrators for an important dynamics with outputs that arises in the vision based [6] landing of aircraft; see (1)-(2) below. A key difficulty is that only imprecise measurements of the two first states are available, so we cannot apply the semi-global or regional stability results mentioned above, nor [13] or its extension in [14]. Thus, we propose a new control, which extends [13] and [14]. Our theory is inspired by forwarding theory in [15].

The controls we construct ensure input-to-state stability with a saturated input. This work improves on the conference version [11] by providing input-to-state stability results that allow measurement delays and sampling [17], [18]. The work [11] only gave a weaker ultimate boundedness result (with no proof) and did not allow measurement delays or sampling.

The notation will be simplified whenever no confusion can arise from the context. The Euclidean norm is denoted by $|\cdot|$, and $|\cdot|_{\mathscr{S}}$ (resp., $|\cdot|_\infty$) denotes the corresponding supremum over any set $\mathscr{S}$ (resp., essential supremum). Given any constant $T>0$, $C_{\rm in}$ denotes the set of all continuous functions $\phi:[-T,0]\to\mathbb{R}^n$, which we call the set of all *initial functions*. We define $\Xi_t\in C_{\rm in}$ by $\Xi_t(s)=\Xi(t+s)$ for all Ξ, $s\le 0$, and $t\ge 0$ for which the equality is defined. For each constant $L>0$, we use the usual saturation function ${\rm sat}_L(x)=\max\{-L,\min\{L,x\}\}$, and ∂ will denote a boundary. Finally, we use the standard definitions of input-to-state stability and class $\mathscr{K}\mathscr{L}$ and $\mathscr{K}_\infty$ functions, e.g., from [7, Chapter 4].

II. Problem statement

We consider the system

$$\dot{x}_1={\rm sat}_{L_1}(x_2),\ \dot{x}_2={\rm sat}_{L_2}(x_3),\ \dot{x}_3={\rm sat}_{L_3}(u), \tag{1}$$

with the state space $\mathbb{R}^3$, the input u valued in $\mathbb{R}$, the L_i's being known positive constants, and the outputs

$$\begin{array}{l} y_1(t)=\eta(t)x_1(\sigma(t))+\delta_1(t)\\ y_2(t)=x_2(\sigma(t))+\delta_2(t),\ \text{and}\ y_3(t)=x_3(t), \end{array} \tag{2}$$

where $\delta=(\delta_1,\delta_2)$ is a vector of unknown piecewise continuous functions for which there are known constants $\underline{\eta}>0$, $\bar{\eta}>\underline{\eta}$, $\bar{\delta}_1\ge 0$ and $\bar{\delta}_2\ge 0$ such that for all $t\ge 0$,

$$\eta(t)\in[\underline{\eta},\bar{\eta}]\ \text{and}\ |\delta_i(t)|\le\bar{\delta}_i\ \text{for}\ i=1,2, \tag{3}$$

and the known piecewise continuous nondecreasing right continuous function $\sigma:\mathbb{R}\to\mathbb{R}$ admits a constant $\bar{\sigma}\ge 0$ such that $t-\bar{\sigma}\le\sigma(t)\le t$ for all $t\ge 0$ and so can model measurement delays and sampling (e.g., by choosing $\sigma(t)=t-\bar{\sigma}$, or $\sigma(t)=t_i$ for all $\in[t_i,t_{i+1})$ for all $i\ge 0$, where $t_0=0$ and the sample times $t_i\ge 0$ admit a constant $\varepsilon_0>0$ such that $\bar{\sigma}\ge t_{i+1}-t_i\ge\varepsilon_0$). The preceding dynamics and outputs agree with the vision based aircraft landing dynamics in [11] except now we allow delays and sampling that were not allowed in [11]; including delays and sampling is strongly motivated by image processing in aircraft systems [1], [8].

Our goal is the input-to-state stabilization of the origin of (1) with a saturating input using only the measurements (2). Since $\eta(t)$ is present in y_1, the classical approaches do not apply. Although (1) is a feedforward system (as defined for instance in [16]), the forwarding approach (as explained, e.g., in [16]) does not apply, except under an additional assumption on the size of $\bar{\eta}-\underline{\eta}$. However, for practical reasons, we cannot impose an assumption of this type.

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Since (1) is not in feedback form, classical backstepping results (e.g., [19]) do not apply. Since η is unknown and not necessarily differentiable, bounded backstepping results in [10] and [12] do not apply. Other approaches (e.g., [5], [9], and [15]) do not apply here either. Moreover, one cannot apply [13], which does not allow uncertain measurements. Thus, we believe the problem solved in this work was open.

III. Statement of main result

Our main result is as follows, where we use the function

$$\begin{aligned} H(\ell,r) \;=\; & \tfrac{5}{2}\bar{\sigma}_* L_2\big\{\ell\bar{\eta}\left(\tfrac{2}{r}+\bar{\sigma}\right)\big[\tfrac{5}{2}\bar{\sigma}_* L_2(1+\ell\bar{\delta}_1) \\ & +2(\bar{\delta}_2+\bar{\sigma}L_2)\big]+\ell\bar{\delta}_1\big\}+2(\bar{\delta}_2+\bar{\sigma}L_2), \end{aligned} \tag{4}$$

where $\bar{\sigma}_* = \max\{\bar{\sigma},1\}$, and where the existence of the required constants ε and λ_a follows because the first inequality in (5a) implies that $2(\bar{\delta}_2+\bar{\sigma}_* L_2) < \frac{5}{2}\bar{\sigma}_* L_2$, and then our condition $H(\lambda_a,r) < \frac{5}{2}\bar{\sigma}_* L_2(1-\varepsilon)$ implies that the argument of the $\tanh^{-1}$ in (7) is in $(0,1)$ and where we assume that the initial functions are constant at the initial time $t_0=0$:

Theorem 1: Assume that the positive constants L_1 and L_2 and the constants $\bar{\sigma}\geq 0$ and $\bar{\delta}_2 \geq 0$ are such that

$$\frac{4\bar{\delta}_2}{L_2} < \bar{\sigma}_* < \frac{L_1(1-e^{-1})^2}{10L_2(1+2e^{-1}+e^{-2})} \tag{5a}$$

$$\text{and } \bar{\delta}_2 \leq \frac{L_1}{4} \tag{5b}$$

hold, where $\bar{\sigma}_* = \max\{\bar{\sigma},1\}$. Choose a constant $r>0$ that is small enough so that $2r^2L_1 < \min\{L_3/2, rL_2\}$. Set $\bar{v} = 3r^2/L_1$. Let $\lambda \in (0,\lambda_a]$ be a constant such that

$$\lambda < \frac{8\underline{\eta}H(\lambda_a,r)}{75b_*\bar{\eta}^2(\bar{\sigma}_*L_2)^2(2/r+\bar{\sigma})(1-\varepsilon)}, \tag{6}$$

where $\lambda_a>0$ and $\varepsilon\in(0,1)$ are constants that are small enough such that $H(\lambda_a,r) < \frac{5}{2}\bar{\sigma}_* L_2(1-\varepsilon)$ and

$$b_* = \tanh^{-1}\left(\frac{2H(\lambda_a,r)}{5\bar{\sigma}_*L_2(1-\varepsilon)}\right). \tag{7}$$

Then we can find functions $\beta\in\mathscr{KL}$ and $\gamma\in\mathscr{K}_\infty$ so that each solution $X:[0,+\infty)\to\mathbb{R}^3$ of (1), in closed loop with

$$\begin{aligned} & u(Y_t) = -ry_3(t)+\mathrm{sat}_{\bar{v}}\big(-r^2y_2(t)-ry_3(t)+\varsigma(y_{1t})\big), \\ & \text{where } \varsigma(y_{1t}) = -\tfrac{r^2}{(1-e^{-1})^2}\big[\phi(y_1(t))-2e^{-1}\phi(y_1(t-\tfrac{1}{r})) \\ & +e^{-2}\phi(y_1(t-\tfrac{2}{r}))\big] \text{ and } \phi(a) = (5/2)\bar{\sigma}_*L_2\tanh(\lambda a), \end{aligned} \tag{8}$$

is such that the input-to-state stability estimate

$$|X(t)| \leq \beta\left(|X|_{[-2/r-\bar{\sigma},0]},t\right)+\gamma\left(|(\bar{\delta}_1,\bar{\delta}_2,\bar{\sigma})|\right) \tag{9}$$

is satisfied for all $t\geq 0$. □

Remark 1: A key feature of (8) is that it only depends on $Y=(y_1,y_2,y_3)$ from (2) and has no distributed terms. Also, η need not be C^1. Theorem 1 is new, even in the significant special case where $\bar{\sigma}=0$, in which case its conclusion is input-to-state stability with respect to $\delta=(\delta_1,\delta_2)$. For fixed L_1 and L_2, (5a)-(5b) constrain the allowable $\bar{\delta}_2$ values. However, there are no constraints on $\bar{\delta}_1$; and for any constants $\bar{\sigma}\geq 0$ and $\bar{\delta}_2\geq 0$, we can compute constants $L_1>0$ and $L_2>0$ such that (5a)-(5b) are satisfied (by choosing L_2 large enough and then choosing L_1 large enough, which produces a restriction on the allowable lower bounds for the allowable L_1 and L_2 for which (5a)-(5b) can be satisfied, which can be expressed in terms of $\bar{\delta}_2$ and $\bar{\sigma}_*$). Our proof of Theorem 1 will show that the theorem remains true if we replace $\bar{\sigma}_* = \max\{\bar{\sigma},1\}$ by $\max\{\bar{\sigma},q\}$ for any constant $q>0$ (and then (5a) shows that larger q's can allow bigger $\bar{\delta}_2$'s for each L_2). We use $\bar{\sigma}_*$ to allow cases where $\bar{\sigma}=0$. We can allow σ in the definition (2) of Y to be unknown, provided we know a constant $\bar{\sigma}\geq 0$ such that $t-\bar{\sigma}\leq\sigma(t)\leq t$ for all $t\geq 0$. □

IV. Technical preliminary results

Our theorem will follow from the following three lemmas, which we prove in the appendices, and where the positivity of the infimum in (13a) follows from (12b) and because our choice of ε implies that $\mathscr{C}$ is a compact neighborhood of 0, and the existence of the $\delta_*>0$ in Lemma 1 is by (12c).

Lemma 1: Let ε, $\bar{d}$, ϕ_s, and r be positive constants and set $\mathfrak{s} = r^2/(1-e^{-1})^2$. Let $\phi:\mathbb{R}\to\mathbb{R}$ be a bounded odd nondecreasing function that is globally Lipschitz, with a global Lipschitz constant $\phi_c>0$. Consider the system

$$\dot{\xi} = \alpha(t,\xi_t)+d(t), \tag{10}$$

where ξ is valued in $\mathbb{R}$, $d:[0,+\infty)\to\mathbb{R}$ is a piecewise continuous function such that $|d(t)|\leq\bar{d}$ for all $t\geq 0$, and

$$\begin{aligned} & \alpha(t,\xi_t) = \\ & -\mathfrak{s}\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\left(\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi\big(\eta(m)\xi(\sigma(m))\big)\mathrm{d}m\right)\mathrm{d}\ell \end{aligned} \tag{11}$$

and σ was defined in Section II. Assume that the conditions

$$\phi_c\bar{\eta}\left(\tfrac{2}{r}+\bar{\sigma}\right)(|\phi|_\infty+\bar{d})+\bar{d} < |\phi|_\infty, \tag{12a}$$

$$\phi(\ell)\geq\phi_s\ell \text{ for all } \ell\in[0,\phi_s], \tag{12b}$$

$$\text{and } \tfrac{1}{2}\left(\tfrac{2}{r}+\bar{\sigma}\right)\mathfrak{p} < \mathfrak{c} \tag{12c}$$

are satisfied, where $\mathfrak{p}$, $\mathfrak{c}$, and $\mathscr{C}$ are defined by

$$\mathfrak{p} = \phi_c^2\bar{\eta}^2 \text{ and } \mathfrak{c} = \inf_{b\in\mathscr{C}\setminus\{0\}}\frac{\phi(\underline{\eta}|b|)}{3|b|}, \text{ and} \tag{13a}$$

$$\begin{aligned} & \mathscr{C} = \\ & \left\{b\in\mathbb{R}:\phi_c\bar{\eta}\left(\tfrac{2}{r}+\bar{\sigma}\right)(|\phi|_\infty+\bar{d})+\bar{d}\geq(1-\varepsilon)\phi(\underline{\eta}|b|)\right\} \end{aligned} \tag{13b}$$

and where the constant $\varepsilon\in(0,1)$ is chosen such that $\phi_c\bar{\eta}(2/r+\bar{\sigma})(|\phi|_\infty+\bar{d})+\bar{d} < (1-\varepsilon)|\phi|_\infty$ and $\bar{\sigma}$ is defined in Section II in terms of the function σ. Set

$$t_0(s) = \max\left\{\frac{4}{r}+\bar{\sigma},\frac{s^2}{2\varepsilon\min\{\xi\phi(\underline{\eta}\xi):\xi\in\partial\mathscr{C}\}}\right\} \tag{14a}$$

$$\begin{aligned} & \text{where } t_\sharp(s) = t_0(s) \\ & +\frac{1}{2\delta_*}\ln\left(\frac{4\underline{\varepsilon}\left(s+t_0(s)(|\phi|_\infty+\bar{d})\right)^2\left(\mathfrak{c}^\flat-0.5(2/r+\bar{\sigma})\mathfrak{p}\right)}{\bar{d}^2\left((2/r+\bar{\sigma})\phi_c\bar{\eta}+1\right)^2}+1\right) \end{aligned} \tag{14b}$$

where $\delta_*>0$ and $\mathfrak{c}^\flat\in(0,\mathfrak{c})$ are constants such that $\delta_* = \mathfrak{c}^\flat-(1/2)\mathfrak{p}(2/r+\bar{\sigma})e^{2\delta_*(4/r+\bar{\sigma})}$ and $(2/r+\bar{\sigma})\mathfrak{p} < 2\mathfrak{c}^\flat$, and $\underline{\varepsilon} = \mathfrak{c}-\mathfrak{c}^\flat$. Then for each initial state $\xi(0)\in\mathbb{R}$, the corresponding solution $\xi:[0,+\infty)\to\mathbb{R}$ of (10) satisfies

$$|\xi(t)| \leq \frac{((2/r+\bar{\sigma})\phi_c\bar{\eta}+1)\bar{d}}{\sqrt{2(\mathfrak{c}-\mathfrak{c}^\flat)(\mathfrak{c}^\flat-0.5(2/r+\bar{\sigma})\mathfrak{p})}} \tag{15}$$

for all $t\geq t_\sharp(|\xi(0)|)$. □

Lemma 2: Consider the two-dimensional system

$$\dot{\lambda}_1 = \mathrm{sat}_{U_1}(\lambda_2),\ \dot{\lambda}_2 = \mathrm{sat}_{U_2}(\rho), \tag{16}$$

where U_1 and U_2 are positive constants and ρ is the input.

Let r, $\bar{v}$, and $\bar{w}$ be positive constants such that

$$\bar{v} < \min\{U_2/2, rU_1\} \text{ and } \bar{w} + r^2\bar{\sigma}U_1 < \bar{v}/2, \tag{17}$$

where σ and $\bar{\sigma}$ are from Section II. Consider (16) with

$$\rho(t) = -r\lambda_2(t) + \text{sat}_{\bar{v}}\big(-r^2\lambda_1(\sigma(t)) - r\lambda_2(t) + w(t)\big) \tag{18}$$

as the input, where w is any piecewise continuous function such that $\sup_{t\geq 0}|w(t)| \leq \bar{w}$, and choose the function

$$t_*(s) = \frac{2s}{\min\{2rU_1, U_2\} - 2\bar{v}} + \frac{(1+r)s}{\bar{v} - 2(\bar{w} + r^2\bar{\sigma}U_1)}. \tag{19}$$

Then for each initial state $\lambda(0) \in \mathbb{R}^2$, the corresponding solution of (16) in closed loop with (18) satisfies

$$\left\{\begin{array}{lll}\dot{\lambda}_1(t) & = & -r\lambda_1(t) + \lambda_3(t) \\ \dot{\lambda}_3(t) & = & -r\lambda_3(t) + w(t) + r^2(\lambda_1(t) - \lambda_1(\sigma(t)))\end{array}\right. \tag{20}$$

for all $t \geq t_*(|\lambda(0)|)$, where $\lambda_3 = \lambda_2 + r\lambda_1$. $\square$

Lemma 3: Let $\bar{\mu} > 0$ and $r > 0$ and $\varepsilon_i > 0$ for $i = 1$ and 2 be constants and $\mu : [0, +\infty) \to [-\bar{\mu}, \bar{\mu}]$ be a piecewise continuous function. Choose $t_2(s) = \frac{s}{\varepsilon_2\bar{\mu}}$ and

$$t_1(s) = t_2(s) + \frac{r}{\varepsilon_1\bar{\mu}(1+\varepsilon_2)}\left(s + \frac{1}{r}t_2(s)\max\left\{s, (1+\varepsilon_2)\frac{\bar{\mu}}{r}\right\}\right).$$

Then for each initial state $z(0) \in \mathbb{R}^2$, the solution $z : [0, +\infty) \to \mathbb{R}^2$ of

$$\dot{z}_1(t) = -rz_1(t) + z_2(t),\ \dot{z}_2(t) = -rz_2(t) + \mu(t) \tag{21}$$

satisfies $|z_2(t)| \leq (1+\varepsilon_2)\bar{\mu}/r$ for all $t \geq t_2(|z_2(0)|)$ and $|z_1(t)| \leq (1+\varepsilon_1)(1+\varepsilon_2)\bar{\mu}/r^2$ for all $t \geq t_1(|z(0)|)$. $\square$

V. Proof of Theorem 1

The proof has two parts. In the first part, we show that our choices of r, λ_a, λ, and ε from the theorem ensure that $\phi(a) = (5/2)\bar{\sigma}_*L_2\tanh(\lambda a)$ satisfies (12a)-(12c) from Lemma 1 with the definitions (13a) and (13b) and the choice $\bar{d} = \phi_c\bar{\delta}_1 + 2(\bar{\delta}_2 + \bar{\sigma}L_2)$ for suitable ϕ_c and ϕ_s, and that

$$\left[1 + 2e^{-1} + e^{-2}\right]|\phi|_\infty \leq \frac{\bar{v}}{6\mathfrak{s}} \tag{22a}$$

$$\text{and } r^2\bar{\sigma}_*L_2 < \frac{\bar{v}}{6} \text{ and } \bar{\delta}_2 \leq \frac{\bar{v}}{6r^2} \tag{22b}$$

hold with $\bar{v} = \frac{3}{2}r^2L_1$ and $\mathfrak{s} = r^2/(1-e^{-1})^2$. In the second part, we combine the first part with our three lemmas.

A. First Part of Proof

Our choice $\phi(a) = (5/2)\bar{\sigma}_*L_2\tanh(\lambda a)$ in the feedback control has the global Lipschitz constant $\phi_c = (5/2)\bar{\sigma}_*L_2\lambda$. Also, our condition $2r^2L_1 < \min\{L_3/2, rL_2\}$ on $r > 0$ from the statement of the theorem ensures that our choice $\bar{v} = \frac{3}{2}r^2L_1$ of $\bar{v}$ satisfies $\bar{v} < \min\{L_3/2, rL_2, 2r^2L_1\}$. Moreover, our conditions (5a)-(5b) ensure that (22a) and (22b) are both satisfied, because $(1-e^{-1})^2/(10(1+2e^{-1}+e^{-2})) < 1/4$; the second inequality in (5a) was used to establish (22a) and the first inequality in (22b), and (5b) implies the second inequality in (22b). Also, since the left side of (12a) with our choices of ϕ and ϕ_c is $H(\lambda, r)$, it follows that (12a) holds for all $\lambda \in (0, \lambda_a)$, because of our choices of λ_a and $\bar{d} = \phi_c\bar{\delta}_1 + 2(\bar{\delta}_2 + \bar{\sigma}L_2)$. Moreover, we can use L'Hopital's Rule to find a constant $\phi_s > 0$ such that (12b) is satisfied. Therefore, the first part of the proof will be complete once we show (12c).

To verify (12c), let $\mathscr{C}_\lambda$ denote the set defined in (13b) (which depends on λ because of the dependence of ϕ on λ). Then $\{\lambda b : b \geq 0, b \in \mathscr{C}_\lambda, \lambda \in (0, \lambda_a]\}$ is a bounded set, since for any nonnegative values $b \in \mathscr{C}_\lambda$ and any $\lambda \in (0, \lambda_a]$, our choice of λ_a from the statement of the theorem gives

$$\begin{aligned}&(5/2)(1-\varepsilon)\bar{\sigma}_*L_2\tanh(\underline{\eta}\lambda b) \\ &\leq \sup_{\lambda\in(0,\lambda_a]}\left[\phi_c\bar{\eta}(2/r+\bar{\sigma})(|\phi|_\infty+\bar{d})+\bar{d}\right] = \sup_{\lambda\in(0,\lambda_a]}H(\lambda, r) \\ &< (1-\varepsilon)|\phi|_\infty = (5/2)(1-\varepsilon)\bar{\sigma}_*L_2,\end{aligned} \tag{23}$$

and $\lim_{b\to+\infty}(1-\varepsilon)\bar{\sigma}_*L_2\tanh(\underline{\eta}\lambda b) = (1-\varepsilon)\bar{\sigma}_*L_2$. Hence,

$$\underline{J} = \inf_{\lambda\in(0,\lambda_a]}\inf_{b\in\mathscr{C}_\lambda\setminus\{0\}}\frac{\phi(\underline{\eta}|b|)}{3|b|\lambda} \tag{24}$$

is a positive constant, and (12c) will be satisfied if

$$\tfrac{1}{2}\left(\tfrac{2}{r} + \bar{\sigma}\right)\bar{\eta}^2\lambda\left(\tfrac{5}{2}\bar{\sigma}_*L_2\right)^2 < \underline{J}. \tag{25}$$

Also, since $\tanh(x)/x$ is decreasing on $(0, +\infty)$, we have

$$\underline{J} = \inf_{\lambda\in(0,\lambda_a]}\inf_{b\in\mathscr{C}_\lambda\setminus\{0\}}\frac{\frac{5}{2}\bar{\sigma}_*L_2\tanh(\lambda\underline{\eta}|b|)}{3|b|\lambda} = \frac{5\bar{\sigma}_*L_2\tanh(\lambda_a\underline{\eta}b_a)}{6b_a\lambda_a}, \tag{26}$$

where b_a is the largest element of $\mathscr{C}_{\lambda_a}$, i.e., $b_a = b_*/(\lambda_a\underline{\eta})$, where b_* is from (7). Hence, (26) gives $\underline{J} = \underline{\eta}H(\lambda_a, r)/(3b_*(1-\varepsilon))$, which we can substitute into the right side of (25) to see that (25) is equivalent to our condition (6) on λ that we assumed in our statement of our theorem. Therefore, (25) and so also (12c) are satisfied.

B. Second Part of Proof

First step. For all $t \geq 0$, our condition (22a) implies that

$$|\varsigma(y_{1t})| \leq \mathfrak{s}|\phi|_\infty + 2\mathfrak{s}e^{-1}|\phi|_\infty + \mathfrak{s}e^{-2}|\phi|_\infty \leq \tfrac{\bar{v}}{6}, \tag{27}$$

and the definition of Y gives $u(Y_t) = -rx_3(t) + \text{sat}_{\bar{v}}(-r^2x_2(\sigma(t)) - rx_3(t) + \varsigma(y_{1t}) - r^2\delta_2(t))$ for all $t \geq 0$. Notice that the second inequality in (22b) and (27) give

$$\sup_{t\geq 0}\left|\varsigma(y_{1t}) - r^2\delta_2(t)\right| \leq \tfrac{\bar{v}}{6} + \tfrac{\bar{v}}{6} = \tfrac{\bar{v}}{3}. \tag{28}$$

We deduce from the first inequality in (22b) that Lemma 2 applies to the (x_2, x_3)-subsystem of (1) in closed-loop with (8) with $\bar{v} = \frac{3}{2}r^2L_1$, $w(t) = \varsigma(y_{1t}) - r^2\delta_2(t)$, $\bar{w} = \bar{v}/3$, and $(U_1, U_2) = (L_2, L_3)$. Hence, with the choice $\mathscr{G}(t) = r^2(x_2(t) - x_2(\sigma(t)))$, there is a continuous increasing t_a such that

$$\left\{\begin{array}{lll}\dot{x}_2(t) & = & -rx_2(t) + x_4(t) \\ \dot{x}_4(t) & = & -rx_4(t) + \varsigma(y_{1t}) - r^2\delta_2(t) + \mathscr{G}(t).\end{array}\right. \tag{29}$$

at all times $t \geq t_a(|X(0)|)$, where $x_4 = x_3 + rx_2$.

From (28) and Lemma 3 (applied with $(z_1, z_2) = (x_2, x_4)$, $\mu(t) = \varsigma(y_{1t}) - r^2\delta_2(t) + \mathscr{G}(t)$, $\bar{\mu} = \bar{v}/3 + r^2\bar{\sigma}L_2$, and small enough positive ε_1 and ε_2) and (22b), we find a continuous increasing t_b such that $t_b(s) \geq t_a(s)$ for all $s \geq 0$ and

$$|x_2(t)| \leq \tfrac{(1+\varepsilon_1)(1+\varepsilon_2)\bar{v}}{2r^2} = (1+\varepsilon_1)(1+\varepsilon_2)\tfrac{3}{4}L_1 < L_1 \tag{30}$$

holds for all $t \geq t_b(|X(0)|)$, since $|\mathscr{G}(t)| \leq r^2L_2\bar{\sigma}$ holds for all $t \geq 0$. We deduce that for all $t \geq t_b(|X(0)|)$, we have

$$\left\{\begin{array}{lll}\dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & -rx_2(t) + x_4(t) \\ \dot{x}_4(t) & = & -rx_4(t) + \varsigma(y_{1t}) - r^2\delta_2(t) + \mathscr{G}(t).\end{array}\right. \tag{31}$$

Second step. Let us introduce these two operators:

$$\Gamma_1(t) = -\mathfrak{s}\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi(y_1(m))\mathrm{d}m\mathrm{d}\ell \tag{32a}$$

$$\begin{aligned}\Gamma_2(t) &= -\mathfrak{s}\int_{t-\frac{1}{r}}^{t} e^{r(m-t)}\phi(y_1(m))\mathrm{d}m \\ &\quad +\mathfrak{s}e^{-1}\int_{t-\frac{2}{r}}^{t-\frac{1}{r}} e^{r(m-t+1/r)}\phi(y_1(m))\mathrm{d}m,\end{aligned} \tag{32b}$$

where $\mathfrak{s} = r^2/(1-e^{-1})^2$ as before. Then simple calculations produce $\dot\Gamma_1(t) = -r\Gamma_1(t)+\Gamma_2(t)$ and $\dot\Gamma_2(t) = -r\Gamma_2(t)+\varsigma(y_{1t})$, by our choice ς from (8). Let $\tilde{x}_2(t) = x_2(t)-\Gamma_1(t)$ and $\tilde{x}_4(t) = x_4(t)-\Gamma_2(t)$. Combining (31) with our formulas for $\dot\Gamma_1$ and $\dot\Gamma_2$, we conclude that

$$\begin{cases} \dot{x}_1(t) = \Gamma_1(t)+\tilde{x}_2(t),\ \dot{\tilde{x}}_2(t) = -r\tilde{x}_2(t)+\tilde{x}_4(t) \\ \dot{\tilde{x}}_4(t) = -r\tilde{x}_4(t)-r^2\delta_2(t)+\mathscr{G}(t) \end{cases} \tag{33}$$

are satisfied for almost all $t \ge t_b(|X(0)|)$.

Third step. Our y_1 formula in (2) and our choice of Γ_1 give

$$\begin{aligned}&\dot{x}_1(t) = \mathfrak{n}(t) \\ &-\mathfrak{s}\textstyle\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi\big(\eta(m)x_1(\sigma(m))\big)\mathrm{d}m\mathrm{d}\ell\end{aligned} \tag{34}$$

for all $t \ge t_b(|X(0)|)$, where

$$\begin{aligned}\mathfrak{n}(t) = \mathfrak{s}\textstyle\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\Big[\phi\big(\eta(m)x_1(\sigma(m)\big) \\ -\phi\Big(\eta(m)x_1(\sigma(m))+\delta_1(m)\Big)\Big]\mathrm{d}m\mathrm{d}\ell+\tilde{x}_2(t).\end{aligned}$$

Since ϕ has the global Lipschitz constant ϕ_c, we have

$$|\mathfrak{n}(t)| \le \mathfrak{s}\phi_c\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}|\delta_1(m)|\mathrm{d}m\mathrm{d}\ell+|\tilde{x}_2(t)|.$$

We can then use Lemma 3 (with the choices $(z_1,z_2) = (\tilde{x}_2,\tilde{x}_4)$, $\mu = -r^2\delta_2+\mathscr{G}(t)$, $\bar\mu = r^2(\bar\delta_2+L_2\bar\sigma)$, $\varepsilon_1 = 1/2$, and $\varepsilon_2 = 1/3$) and the boundedness of Γ_1 to find a continuous increasing t_c such that $t_c(s) \ge t_b(s)$ for all $s \ge 0$ and such that $|\tilde{x}_2(t)| \le 2(\bar\delta_2+\bar\sigma L_2)$ and $|\tilde{x}_4(t)| \le \frac{4}{3}r(\bar\delta_2+\bar\sigma L_2)$ hold for all $t \ge t_c(|X(0)|)$. Hence, our choice of $\mathfrak{s}$ implies that $|\mathfrak{n}(t)| \le \phi_c\bar\delta_1+2(\bar\delta_2+\bar\sigma L_2)$ for all $t \ge t_c(|X(0)|)$. Since ϕ satisfies the requirements of Lemma 1 with $\bar{d} = \phi_c\bar\delta_1+2(\bar\delta_2+\bar\sigma L_2)$, we can apply Lemma 1 to the x_1 subsystem (34) to obtain a continuous increasing t_d such that (i) $t_d(s) \ge t_c(s)$ for all $s \ge 0$ and (ii) for all $t \ge t_d(|X(0)|)$, the following is satisfied:

$$|x_1(t)| \le \frac{(\phi_c\bar\eta(2/r+\bar\sigma)+1)(\phi_c\bar\delta_1+2(\bar\delta_2+\bar\sigma L_2))}{\sqrt{2(\mathfrak{c}-\mathfrak{c}^\flat)(\mathfrak{c}^\flat-0.5(2/r+\bar\sigma)\mathfrak{p})}} \tag{35}$$

Final step. Our formulas $\tilde{x}_2(t) = x_2(t)-\Gamma_1(t)$ and $\tilde{x}_4(t) = x_4(t)-\Gamma_2(t)$ and our choice of t_c imply that

$$\begin{aligned}&|x_4(t)| \le \tfrac{4r}{3}(\bar\delta_2+\bar\sigma L_2)+|\Gamma_2(t)| \text{ and} \\ &|x_2(t)| \le 2(\bar\delta_2+\bar\sigma L_2)+|\Gamma_1(t)|\end{aligned} \tag{36}$$

hold for all $t \ge t_d(|X(0)|)$. From the definition of x_4, we deduce that $x_3 = x_4-rx_2$, so for all $t \ge t_d(|X(0)|)$,

$$\begin{aligned}|x_2(t)| &\le 2(\bar\delta_2+\bar\sigma L_2)+|\Gamma_1(t)| \text{ and} \\ |x_3(t)| &\le \tfrac{10}{3}r(\bar\delta_2+\bar\sigma L_2)+|\Gamma_2(t)|+r|\Gamma_1(t)|.\end{aligned} \tag{37}$$

Next note that the properties of ϕ and η ensure that

$$|\Gamma_1(t)| \le \mathfrak{s}\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi_c(\bar\eta|x_1(\sigma(m))|+\bar\delta_1)\mathrm{d}m\mathrm{d}\ell.$$

Therefore, when $t \ge t_d(|X(0)|)+2/r+\bar\sigma$, we have

$$\begin{aligned}&|\Gamma_1(t)| \le \\ &\phi_c\left(\bar\eta\,\frac{\big(\phi_c\bar\eta(2/r+\bar\sigma)+1\big)\big(\phi_c\bar\delta_1+2(\bar\delta_2+\bar\sigma L_2)\big)}{\sqrt{2(\mathfrak{c}-\mathfrak{c}^\flat)(\mathfrak{c}^\flat-0.5(2/r+\bar\sigma)\mathfrak{p})}}+\bar\delta_1\right),\end{aligned} \tag{38}$$

by our choice $\mathfrak{s} = r^2/(1-e^{-1})^2$ and (35). For similar reasons,

$$\begin{aligned}&|\Gamma_2(t)| \le \mathfrak{s}\textstyle\int_{t-\frac{1}{r}}^{t} e^{r(m-t)}\phi_c\left(\bar\eta|x_1(\sigma(m))|+\bar\delta_1\right)\mathrm{d}m \\ &+\mathfrak{s}e^{-1}\textstyle\int_{t-\frac{2}{r}}^{t-\frac{1}{r}} e^{r(m-t+1/r)}\phi_c\left(\bar\eta|x_1(\sigma(m))|+\bar\delta_1\right)\mathrm{d}m\end{aligned} \tag{39}$$

and one can use (35) to prove that for all $t \ge t_d(|X(0)|)+2/r+\bar\sigma$, we have

$$\begin{aligned}&|\Gamma_2(t)| \le \\ &\tfrac{r\left(1+e^{-1}\right)\phi_c}{1-e^{-1}}\left(\frac{\bar\eta\left((2/r+\bar\sigma)\phi_c\bar\eta+1\right)\left(\phi_c\bar\delta_1+2(\bar\delta_2+\bar\sigma L_2)\right)}{\sqrt{2(\mathfrak{c}-\mathfrak{c}^\flat)(4\mathfrak{c}^\flat-0.5(2/r+\bar\sigma)\mathfrak{p})}}+\bar\delta_1\right).\end{aligned} \tag{40}$$

If we now use the right sides of (38) and (40) to upper bound $|\Gamma_1(t)|$ and $|\Gamma_2(t)|$ from the right sides of (37) and then combine the results with (35), then we obtain a function $\gamma_0 \in \mathscr{K}_\infty$ such that $|X(t)| \le \gamma_0(|(\bar\delta_1,\bar\delta_2,\bar\sigma)|)$ for all $t \ge t_d(|X(0)|)+2/r+\bar\sigma$. We can also use Gronwall's inequality (applied to $|X|_{[t-2/r-\bar\sigma,t]}$) and the linear growth of the right side of the closed loop system given by (1) and (8) to obtain a constant $\bar{G} > 0$ such that $|X(t)| \le \bar{G}\big(|X|_{[-2/r-\bar\sigma,0]}+|(\bar\delta_1,\bar\delta_2)|^2\big)$ for all $t \in [0, t_d(|X(0)|)+2/r+\bar\sigma]$, since we can choose T_d to have the affine form $T_d(s) = \bar{T}(s+1)$ for some constant $\bar{T} > 0$ and then apply the triangle inequality to $|x(0)||(\bar\delta_1,\bar\delta_2)|$). Hence, we can choose $\beta(s,t) = \bar{G}e^{t_d(s)+2/r+\bar\sigma-t}s$ and $\gamma(s) = \gamma_0(s)+\bar{G}s^2$ to produce the required input-to-state stability estimate.

VI. Conclusions

We adapted a recent backstepping strategy to solve a difficult problem of stabilizing a chain of three saturating integrands with imprecise output measurements, using artificial delays, meaning the control can contain both current and time lagged output measurements, even when current output measurements are available. This made it possible to prove input-to-state stability under saturated controls, sampling, and measurement delays. In future work, we hope to develop ways to find maximum allowable values for the parameter λ in Theorem 1 (to obtain least conservative results for the amplitude of the control), to allow input delays, and to apply this work to aircraft models from [11] of the form (1) which produce the outputs (2) when image processing is present.

Appendix 1: Proof of Lemma 1

The finite escape time phenomenon does not occur for (10), since α is bounded. Choose $V(\xi) = \frac{1}{2}\xi^2$ and $W(\xi) = \xi\phi(\underline{\eta}\xi)$. Then W is positive definite, by our assumptions on ϕ. Also, along all solutions of (10), we have

$$\begin{aligned}&\dot{V}(t) = -\mathfrak{s}\xi(t)\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi(\eta(m)\xi(t))\mathrm{d}m\mathrm{d}\ell \\ &+\mathfrak{s}\xi(t)\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\Big[\phi(\eta(m)\xi(t)) \\ &-\phi(\eta(m)\xi(\sigma(m)))\Big]\mathrm{d}m\mathrm{d}\ell+\xi(t)d(t) \text{ for all } t \ge 0,\end{aligned}$$

so the global Lipschitz constant ϕ_c for ϕ gives

$$\dot V(t) \le -\mathfrak{s}\xi(t)\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi(\eta(m)\xi(t))\mathrm{d}m\mathrm{d}\ell$$
$$+\mathfrak{s}\phi_c|\xi(t)|\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}|\eta(m)|$$
$$\times|\xi(t)-\xi(\sigma(m))|\mathrm{d}m\mathrm{d}\ell+\xi(t)d(t)$$

along all solutions of (10) for all $t \ge 0$. From the other properties of ϕ and our choice $\mathfrak{s} = r^2/(1-e^{-1})^2$, we get

$$\begin{aligned}\dot V(t) &\le -\mathfrak{s}\xi(t)\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\phi(\underline{\eta}\xi(t))\mathrm{d}m\mathrm{d}\ell\\ &\quad+\mathfrak{s}\phi_c|\xi(t)|\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\bar\eta\\ &\quad\times|\xi(t)-\xi(\sigma(m))|\mathrm{d}m\mathrm{d}\ell+\xi(t)d(t)\\ &= -W(\xi(t))+\mathfrak{s}\phi_c\bar\eta|\xi(t)|\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\\ &\quad\times|\xi(t)-\xi(\sigma(m))|\mathrm{d}m\mathrm{d}\ell+\xi(t)d(t).\end{aligned}$$

Next note that $|\xi(\sigma(m))-\xi(t)| \le \int_{\sigma(m)}^{t}|\dot\xi(s)|\mathrm{d}s$ holds for all $m\in\mathbb{R}$ and $t\in\mathbb{R}$ such that $0\le m\le t$. It follows that

$$\begin{aligned}\dot V(t) &\le -W(\xi(t))\\ &\quad+\mathfrak{s}\phi_c\bar\eta|\xi(t)|\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\\ &\quad\times\int_{m-\bar\sigma}^{t}|\dot\xi(s)|\mathrm{d}s\mathrm{d}m\mathrm{d}\ell+\xi(t)d(t) \qquad \text{(A.1)}\\ &\le -W(\xi(t))+\phi_c\bar\eta|\xi(t)|\int_{t-2/r-\bar\sigma}^{t}|\dot\xi(s)|\mathrm{d}s\\ &\quad+\xi(t)d(t)\end{aligned}$$

for all $t \ge 2/r$, where the last inequality used the bound $\int_{m-\bar\sigma}^{t}|\dot\xi(s)|\mathrm{d}s \le \int_{t-2/r-\bar\sigma}^{t}|\dot\xi(s)|\mathrm{d}s$ for all $m\in[t-2/r,t]$. Hence, $\dot V(t) \le -\varepsilon W(\xi(t)) + [(2/r+\bar\sigma)\phi_c\bar\eta(|\phi|_\infty+\bar d)+\bar d-(1-\varepsilon)\phi(\underline{\eta}|\xi(t)|)]|\xi(t)|$, which follows because $|\dot\xi(t)| \le |\phi|_\infty+\bar d$ (by our choice of $\mathfrak{s}$). Therefore, at all times $t \ge 2/r$ when $(2/r+\bar\sigma)\phi_c\bar\eta(|\phi|_\infty+\bar d)+\bar d \le (1-\varepsilon)\phi(\underline{\eta}|\xi(t)|)$, we have

$$\dot V(t) \le -\varepsilon W(\xi(t)). \qquad \text{(A.2)}$$

We next show that $|\xi(t)|\in\mathscr{C}$ for all $t\ge t_0(|\xi(0)|)$, where t_0 is defined by (14a) and $\mathscr{C}$ is defined in (13b), by considering two cases. First consider the case where $|\xi(0)|\in\mathscr{C}$. In that case, (A.2) and the positive definiteness of W imply that $|\xi(t)|\in\mathscr{C}$ for all $t\ge 0$. Next consider the case where $|\xi(0)|\notin\mathscr{C}$. Then for all t such that $|\xi(\ell)|\notin\mathscr{C}$ for all $\ell\in[0,t]$, our decay estimate (A.2) and our choice of W imply that $\dot V(t)\le -\varepsilon\min\{W(b): b\in\partial\mathscr{C}\}$ and so also $0\le V(\xi(t))\le V(\xi(0))-\varepsilon\min\{W(b): b\in\partial\mathscr{C}\}t$, which gives $t\le t_0(|\xi(0)|)$, so $|\xi(t)|\in\mathscr{C}$ for all $t\ge t_0(|\xi(0)|)$.

Then (A.1) and the definition of $\mathfrak{c}$ in (13a) give

$$\dot V(t) \le -3\mathfrak{c}\xi^2(t)+\phi_c\bar\eta|\xi(t)|\int_{t-2/r-\bar\sigma}^{t}|\dot\xi(s)|\mathrm{d}s+\xi(t)d(t)$$

for all $t\ge t_0(|\xi(0)|)$ and

$$|\dot\xi(t)| \le$$
$$|d(t)|+\phi_c\bar\eta\mathfrak{s}\int_{t-\frac{1}{r}}^{t} e^{r(\ell-t)}\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}|\xi(\sigma(m))|\mathrm{d}m\mathrm{d}\ell$$

for all $t\ge 0$. We deduce that for all $t\ge t_0(|\xi(0)|)$,

$$\begin{aligned}\dot V(t) &\le -3\mathfrak{c}\xi^2(t)+\phi_c\bar\eta|\xi(t)|\int_{t-\frac{2}{r}-\bar\sigma}^{t}\Big[\phi_c\bar\eta\mathfrak{s}\int_{s-\frac{1}{r}}^{s} e^{r(\ell-s)}\\ &\quad\times\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}|\xi(\sigma(m))|\mathrm{d}m\mathrm{d}\ell+|d(s)|\Big]\mathrm{d}s\\ &\quad+\xi(t)d(t)\\ &\le -3\mathfrak{c}\xi^2(t)+\Omega(t)|\xi(t)|+\mathfrak{p}\mathfrak{s}\int_{t-\frac{2}{r}-\bar\sigma}^{t}\int_{s-\frac{1}{r}}^{s} e^{r(\ell-s)}\\ &\quad\times\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}|\xi(t)||\xi(\sigma(m))|\mathrm{d}m\mathrm{d}\ell\mathrm{d}s,\end{aligned}$$

where $\mathfrak{p}$ is defined in (13a) and

$$\Omega(t) = \phi_c\bar\eta\int_{t-2/r-\bar\sigma}^{t}|d(s)|\mathrm{d}s+|d(t)|.$$

Using $|\xi(t)||\xi(\sigma(m))| \le \frac{1}{2}\xi^2(t)+\frac{1}{2}\xi^2(\sigma(m))$, we obtain

$$\begin{aligned}\dot V(t) &\le \left[-3\mathfrak{c}+\tfrac{1}{2}(2/r+\bar\sigma)\phi_c^2\bar\eta^2\right]\xi^2(t)\\ &\quad+\Omega(t)|\xi(t)|+\tfrac{1}{2}\mathfrak{p}\mathfrak{s}\int_{t-2/r-\bar\sigma}^{t}\int_{s-1/r}^{s} e^{r(\ell-s)}\\ &\quad\times\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\xi^2(\sigma(m))\mathrm{d}m\mathrm{d}\ell\mathrm{d}s\\ &\le -\mathfrak{c}\xi^2(t)+\tfrac{\mathfrak{p}\mathfrak{s}}{2}\int_{t-2/r-\bar\sigma}^{t}\int_{s-1/r}^{s} e^{r(\ell-s)}\\ &\quad\times\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\xi^2(\sigma(m))\mathrm{d}m\mathrm{d}\ell\mathrm{d}s+\Omega(t)|\xi(t)|\end{aligned}$$

for all $t\ge t_0(|\xi(0)|)$, where the last inequality used (12c). Then with the choice $\underline{\varepsilon} = \mathfrak{c}-\mathfrak{c}^\flat$, we can use Holder's inequality to get $\Omega(t)|\xi(t)| \le \underline{\varepsilon}\xi^2(t)+\frac{1}{4\underline{\varepsilon}}\Omega^2(t)$ and so also

$$\begin{aligned}&\dot V(t) \le -\mathfrak{c}^\flat\xi^2(t)+\tfrac{\mathfrak{p}\mathfrak{s}}{2}\int_{t-\frac{2}{r}-\bar\sigma}^{t}\int_{s-\frac{1}{r}}^{s} e^{r(\ell-s)}\\ &\quad\times\int_{\ell-\frac{1}{r}}^{\ell} e^{r(m-\ell)}\xi^2(\sigma(m))\mathrm{d}m\mathrm{d}\ell\mathrm{d}s+\frac{1}{4\underline{\varepsilon}}\Omega^2(t)\end{aligned} \qquad \text{(A.3)}$$

for all $t\ge t_0(|\xi(0)|)$. Since $\Omega(t)\le((2/r+\bar\sigma)\phi_c\bar\eta+1)\bar d$, we deduce that $\dot V(t)\le -2\mathfrak{c}^\flat V(\xi(t))+(2/r+\bar\sigma)\mathfrak{p}\sup\{V(\xi(a)): a\in[t-(4/r)-\bar\sigma,t]\}+\Delta$, where $\Delta = ((2/r+\bar\sigma)\phi_c\bar\eta+1)^2\bar d^2/(4\underline{\varepsilon})$. Then a variant of the usual Halanay's inequality proof (e.g., from [4, Section 4.1.2]) implies that

$$\begin{aligned}&V(\xi(t)) \le \tfrac{\Delta}{2(\mathfrak{c}^\flat-0.5(2/r+\bar\sigma)\mathfrak{p})}+\\ &e^{-2\delta_*(t-t_0(|\xi(0)|))}\sup_{-(4/r+\bar\sigma)\le\theta\le 0}V(\xi(t_0(|\xi(0)|)+\theta))\end{aligned} \qquad \text{(A.4)}$$

holds for all $t\ge t_0(|\xi(0)|)$. (The proof of the required variant of Halanay's inequality is obtained from the proof of [4, Lemma 4.2] with the choices $\delta_0=\mathfrak{c}^\flat$, $\delta_1 = 0.5(2/r+\bar\sigma)\mathfrak{p}$, and Δ added to the upper bound on the right side of $\dot V$, by choosing the comparison function $y = ke^{-2\delta_*(t-t_0(|\xi(0)|))}+\frac{\Delta}{2(\delta_0-\delta_1)}$ where k is the supremum in (A.4).) Therefore,

$$\begin{aligned}&V(\xi(t)) \le \tfrac{\Delta}{2(\mathfrak{c}^\flat-0.5(2/r+\bar\sigma)\mathfrak{p})}\\ &+\tfrac{1}{2}e^{-2\delta_*(t-t_0(|\xi(0)|))}\big(|\xi(0)|+t_0(|\xi(0)|)(|\phi|_\infty+\bar d)\big)^2\end{aligned} \qquad \text{(A.5)}$$

holds for all $t\ge t_0(|\xi(0)|)$. Also, (15) can be written as

$$V(\xi(t)) \le \tfrac{\Delta}{\mathfrak{c}^\flat-0.5(2/r+\bar\sigma)\mathfrak{p}}. \qquad \text{(A.6)}$$

The lemma now follows because the right side of (A.5) will be bounded above by the right side of (A.6) if $t\ge t_\sharp(|\xi(0)|)$.

Appendix 2: Proof of Lemma 2

Set $v(t) = \mathrm{sat}_{\bar{v}}(-r^2\lambda_1(\sigma(t)) - r\lambda_2(t) + w(t))$. Then $\dot{\lambda}_2 = \mathrm{sat}_{U_2}(-r\lambda_2(t) + v(t))$ and $|v(t)| \le \bar{v}$ for all $t \ge 0$. Hence, for any $\varepsilon > 0$ such that $\varepsilon\bar{v} < U_2$, there is a $t_\varepsilon \ge 0$ such that

$$|\lambda_2(t)| \le (1+\varepsilon)\bar{v}/r \tag{A.7}$$

for all $t \ge t_\varepsilon$, namely, $t_\varepsilon = |\lambda_2(0)|/(\varepsilon\bar{v})$, where we omit the dependence of t_ε on $|\lambda_2(0)|$ to keep our notation simple. This is because if t is such that $\lambda_2(s) > (1+\varepsilon)(\bar{v}/r)$ (resp., $\lambda_2(s) < -(1+\varepsilon)\bar{v}/r$) holds for all $s \in [0,t]$, then $\dot{\lambda}_2(s) < -\varepsilon\bar{v}$ (resp., $\dot{\lambda}_2(s) > \varepsilon\bar{v}$) for all $s \in [0,t]$ and $0 \le \lambda_2(t) \le \lambda_2(0) - \varepsilon\bar{v}t$ (resp., $0 \ge \lambda_2(t) \ge \lambda_2(0) + \varepsilon\bar{v}t$), and because if (A.7) holds at some $t \ge 0$, then the structure of the system $\dot{\lambda}_2 = \mathrm{sat}_{U_2}(-r\lambda_2(t) + v(t))$ implies that (A.7) also holds for all later times. Hence, $\sup_{t \ge t_\varepsilon} |-r\lambda_2(t) + v(t)| \le (2+\varepsilon)\bar{v}$. If we now specialize the preceding argument to $\varepsilon = \min\{(U_2/(2\bar{v})) - 1, (rU_1/\bar{v}) - 1\}$, then we get $|-r\lambda_2(t) + v(t)| \le (1+\varepsilon)\bar{v} + \frac{U_2}{2} \le U_2$ and $|\lambda_2(t)| \le U_1$ for all

$$t \ge \frac{|\lambda_2(0)|}{\varepsilon\bar{v}} = \frac{2|\lambda_2(0)|}{\min\{2rU_1, U_2\} - 2\bar{v}}. \tag{A.8}$$

Hence, for all $t \ge t_\varepsilon$, we have $\dot{\lambda}_1(t) = \lambda_2(t)$ and $\dot{\lambda}_2(t) = -r\lambda_2(t) + \mathrm{sat}_{\bar{v}}(-r^2\lambda_1(\sigma(t)) - r\lambda_2(t) + w(t))$. Choosing $\lambda_3 = \lambda_2 + r\lambda_1$ and $w^\sharp(t) = w(t) + r^2(\lambda_1(t) - \lambda_1(\sigma(t)))$ now gives

$$\begin{cases} \dot{\lambda}_1(t) &= -r\lambda_1(t) + \lambda_3(t) \\ \dot{\lambda}_3(t) &= \mathrm{sat}_{\bar{v}}(-r\lambda_3(t) + w^\sharp(t)) \end{cases} \tag{A.9}$$

for all $t \ge t_\varepsilon$. Since $|w(t)| \le \bar{w}$ and $|\dot{\lambda}_1(t)| \le U_1$ hold for all $t \ge 0$, we get $|w^\sharp(t)| \le \bar{w} + r^2\bar{\sigma}U_1 < \bar{v}/2$ for all $t \ge 0$, so with the choice $\varepsilon_* = (\bar{v}/\bar{w}^\sharp) - 2$, we have $|\lambda_3(t)| \le (1+\varepsilon_*)\bar{w}^\sharp/r$ for all $t \ge t_\varepsilon + |\lambda_3(0)|/(\varepsilon_*\bar{w}^\sharp)$, where $\bar{w}^\sharp = \bar{w} + r^2\bar{\sigma}U_1$ (which follows by replacing U_2, v, $\bar{v}$, and ε in the argument from the first part of the proof by $\bar{v}$, $w^\sharp$, $\bar{w}^\sharp$, and ε_* respectively, and using the second inequality in (17) to ensure that $\varepsilon_* > 0$) and so also $|-r\lambda_3(t) + w^\sharp(t)| \le (2+\varepsilon_*)\bar{w}^\sharp = \bar{v}$. The conclusion of the lemma follows from the bound $|\lambda_3| \le (1+r)|\lambda|$.

Appendix 3: Proof of Lemma 3

The assertion about $|z_2(t)|$ is shown by the same argument we gave for the bound on $|\lambda_2(t)|$ in Appendix 2, except with λ_2, ε, and $\bar{v}$ replaced by z_2, ε_2, and $\bar{\mu}$, respectively. To prove the assertion about $|z_1(t)|$, first notice that if $|z_1(t)| \le (1+\varepsilon_1)(1+\varepsilon_2)\bar{\mu}/r^2$ holds at a time $t_* \ge t_2(|z_2(0)|)$, then this inequality holds for all $t \ge t_*$, because $\dot{z}_1(t) < 0$ (resp., $\dot{z}_1(t) > 0$) holds at all times where $|z_1(t)| > (1+\varepsilon_1)(1+\varepsilon_2)\bar{\mu}/r^2$ when $z_1(t) > 0$ (resp., $z_1(t) < 0$). If $z_1(t) > (1+\varepsilon_1)(1+\varepsilon_2)\frac{\bar{\mu}}{r^2}$ holds for all t on some interval $[t_2(|z_2(0)|), s]$ for some s, then, for all t in this interval,

$$\begin{aligned} &0 \le z_1(t) \le z_1(t_2(|z_2(0)|)) \\ &- \left(\tfrac{r(1+\varepsilon_1)(1+\varepsilon_2)\bar{\mu}}{r^2} - (1+\varepsilon_2)\tfrac{\bar{\mu}}{r}\right)\left(t - t_2(|z_2(0)|)\right) \\ &\le |z_1(t_2(|z_2(0)|))| - \varepsilon_1(1+\varepsilon_2)\tfrac{\bar{\mu}}{r}\left(t - t_2(|z_2(0)|)\right). \end{aligned} \tag{A.10}$$

Moreover, $|z_2(\ell)| \le \max\{|z_2(0)|, (1+\varepsilon_2)\bar{\mu}/r\}$ holds for all $\ell \ge 0$, so variation of parameters gives

$$\begin{aligned} &|z_1(t_2(|z_2(0)|))| \le \\ &|z_1(0)| + \tfrac{1}{r}t_2(|z_2(0)|)\max\left\{|z_2(0)|, (1+\varepsilon_2)\tfrac{\bar{\mu}}{r}\right\}. \end{aligned} \tag{A.11}$$

Hence, we can combine (A.10)-(A.11) to get

$$s \le t_2(|z_2(0)|) + \tfrac{r}{\varepsilon_1(1+\varepsilon_2)\bar{\mu}}|z_1(t_2(|z_2(0)|))| \le t_2(|z_2(0)|) + \tfrac{r}{\varepsilon_1(1+\varepsilon_2)\bar{\mu}}\left(|z_1(0)| + \tfrac{1}{r}t_2(|z_2(0)|)\max\left\{|z_2(0)|, (1+\varepsilon_2)\tfrac{\bar{\mu}}{r}\right\}\right).$$

Analogous arguments for the case where $z_1(t) < -(1+\varepsilon_1)(1+\varepsilon_2)\frac{\bar{\mu}}{r^2}$ holds for all times t on some interval $[t_2(|z_2(0)|), s]$ now complete the proof of the lemma.

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